Lecture 31 - SHM II and waves

The pendulum

Simple harmonic motion of a mass/spring system occurs because of the relationship between the displacement of the spring and the restoring force, ie Hooke's law $F = ma = -kx$. Any system where the acceleration a is proportional to the negative of the displacement ($-x$ in this problem) shows simple harmonic motion. We shall see that the pendulum has this property, except that we use polar co-ordinates instead of Cartesian co-ordinates.

Consider a mass m at the end of a massless string of length L in a gravitational field producing gravitational acceleration g. The string is attached to a rigid support and the mass oscillates freely without friction. If the mass is moved through an angle θ from its equilibrium position and released the mass oscillates and the oscillations obey the laws of simple harmonic motion. Note that the angle θ is measured from the equilibrium position, which is vertical. The equation describing the time dependence of the angle is,

$$\theta(t) = \theta_0 Cos(\omega t + \delta). \tag{1}$$

The length of the arc from the equilibrium position is related to the angle through $s(t) = L\theta(t)$, so the equation for the arclength as a function of time is,

$$s(t) = L\theta_0 Cos(\omega t + \delta) \tag{2}$$

while the tangential velocity is given by,

$$v(t) = -v_0 Sin(\omega t + \delta) \tag{3}$$

and the tangential acceleration is given by,

$$a(t) = -a_0 Cos(\omega t + \delta) \tag{4}$$

To find the equation for the angular frequency, ω (not that this is NOT the same as the angular velocity in this problem - this is confusing but important), we need to find the force acting on the mass at arclength $s(t)$ from the equilibrium position. The relation between force and displacement for the pendulum is given by,

$$F_t = -mgSin(\theta) \approx -mg\theta = -\frac{mg}{L}s(t) = ma \tag{5}$$

From this equation we see that the restoring force for pendulum motion is $F_t = -(mg/L)s$, which looks just like that for the spring, provided we make the replacement $k \to mg/L$. In terms of acceleration, we have $a = -gs/L = -\omega^2 s$. The frequency of oscillation of the pendulum is then given by,

$$\omega = (\frac{g}{L})^{1/2}; \quad \text{and the period is} \quad T = 2\pi(\frac{L}{g})^{1/2} \tag{6}$$

The energy stored in the pendulum is a sum of the kinetic and potential energies, $KE+PE$. The kinetic energy is $mv^2/2$ as usual, while the potential energy is given by,

$$PE = mgh = mgL(1 - cos(\theta)) \tag{7}$$

The potential energy is a maximum when the pendulum is at the maximum angle, while the kinetic energy is maximum when the pending moves through its equilibrium position.

The pendulum formulas can be adapted to extended bodies which may also undergo SHM, with a frequency given by,

$$\omega = (\frac{mgL}{I})^{1/2} \tag{8}$$

Note that the moment of inertia I in this formula is about the pivot point of the pendulum, ie where it is attached to the rigid support and L is the distance from the support to the center of mass of the object. It is easy to show that this reduces to the case of a point mass by using $I = mL^2$ for that case.

When there is friction or damping, the SHM oscillations are damped and eventually die out. If the damping is so strong that the oscillations never occur, the system is called overdamped, while in the other case where damping is weak the system is called underdamped.

Travelling Waves

Travelling waves are described by a SHM at each point on the wave. At each point waves still have a frequency or period, which describes their periodicity in time. However waves also have a wavelength which determines how often they repeat in space. We therefore need to introduce two new quantities to describe a wave, the wave velocity, v, and the wavelength λ.

To find the relation between T, λ and v, choose a point on the travelling wave, say a creast at a given time. During a period after this time, the crest decreases and then returns to its peak value. During this period a wavelength of the wave passes beneath our reference point. The velocity of the wave is then,

$$v = \frac{\lambda}{T} = f\lambda \tag{9}$$

This is the most important equation in wave motion. Note that there are several ways quantities related to the wavelength and containing the same physics, for example the wavenumber is $k = 2\pi/\lambda$ is often used instead of the wavelength.

There are two broad classes of waves, transverse waves and longitudinal waves. The most familiar wave is the transverse wave, such as waves on a string and waves at the surface of liquids such as water. They are called transverse waves because the displacements are perpendicular or transverse to the direction of wave motion. The height of a transverse travelling wave is given by,

$$y(x,t) = ACos(x - vt) = ACos(kx - wt) \tag{10}$$

This is very general. For each situation where a wave occurs, the frequency, wavelength and velocity are related to the properties of the material, as we shall see. Radio waves, light and other EM radiation are also transverse waves. An alternative type of wave is a longitudinal wave. The most notable example of this type of wave is a sound wave. Sound waves are really oscillations in the pressure inside the material. Sound waves can occur in gases, liquids and in solids and in each case the sound velocity is different as we shall see in Chapter 14.

As an example of a transverse wave, consider the waves on a string, like a guitar or a rope. The wavespeed v is given by,

$$v = (\frac{F}{\mu})^{1/2} \tag{11}$$

where F is the tension is the string or wire and $\mu = mass/length$ which is the mass per unit length of the string. The higher the tension and the lower the mass per unit length then the higher the wavespeed. For fixed wavelength, this implies higher frequency sound as we experience for example with a guitar.

Adding waves: Interference

When two waves meet, they interfere and two cases illustrate what can happen. If two waves have crests at the same places at the same time, they add and this is called constructive interference, and the waves are said to be “in phase”. If two waves have are out of phase, so that one wave has a maximum just where the other has a minimum, the two waves interfere destructively. In fact if the two waves have exactly the same amplitude, they annihilate each other. The property of interference is extremely important and devices such as x-rays, lens’s, microscopes etc rely on understanding and controlling wave interference. We shall look at these phenomena in more detail using sound waves as an example

Reflection of Waves

When a wave hits a wall, it can reflect. If the wall is hard, then the wave inverts on reflection. If the rope can move at the reflection point, then the wave is reflected without inversion.